Dense CRF

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In image segmentation, a random field $\mathbf{I} = \{I_1, I_2, \cdots, I_N\}$ corresponds to the set of all image pixels $i \in \mathcal{V} = \{1, 2, \cdots, N\}$. each pixel i is associated with a finite set of possible labels $\mathcal{L} = \{l_1, l_2, \cdots, l_L\}$ modeled by a variable $X_i \in \mathcal{L}$, I_i is the color vector of pixel i and X_i is the label assigned to pixel i. A conditional random field $(\mathbf{I}, \mathbf{X})$ is characterized by a Gibbs distribution: $P(\mathbf{X}|\mathbf{I}) = \frac{1}{Z(\mathbf{I})} \exp\left\{-\sum_{c \in \mathcal{C}_\mathcal{G}} \phi_c(\mathbf{X}_c|\mathbf{I})\right\}$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph on $\mathbf{X}$ and each clique c in a set of cliques $\mathcal{C}_\mathcal{G}$ in $\mathcal{G}$ induces a potential ϕ_c. The Gibbs energy of a labeling $\mathbf{x} \in \mathcal{L}^N$ is $E(\mathbf{x}|\mathbf{I}) = \sum_{c \in \mathcal{C}_\mathcal{G}} \phi_c(\mathbf{x}_c|\mathbf{I})$. The maximum a posteriori (MAP) labeling of the random field is $\mathbf{x}^* = \arg\max_{\mathbf{x} \in \mathcal{L}^N} P(\mathbf{x}|\mathbf{I})$.

In the fully connected pairwise CRF model, $\mathcal{G}$ is a complete graph on $\mathbf{X}$ and $\mathcal{C}_\mathcal{G}$ is the set of all unary and pairwise cliques. The corresponding Gibbs energy is

$$E(\mathbf{x}|\mathbf{I}) = \sum_{i=1}^{N} \left[\psi_u(x_i) + \sum_{i<j} \psi_p(x_i, x_j)\right]$$

$$\psi_p(x_i, x_j) = \mu(x_i, x_j) \left[w_b \exp\left(-\frac{|p_i - p_j|}{2\alpha^2} - \frac{|I_i - I_j|}{2\beta^2}\right) + w_s \exp\left(-\frac{|p_i - p_j|}{2\gamma^2}\right)\right]$$

where $\psi_u(\cdot)$ is the unary potential describes the cost of the pixel assigning the corresponding label, $\psi_p(\cdot)$ is the pairwise potential which encourages similar pixels to have the same label assignments. $\mu(x_i, x_j)$ is a label compatibility function which introduces a penalty for nearby similar pixels that are assigned different labels. w_s and w_b are the spatial kernel weight and bilateral kernel weight, respectively. α, β, and γ are the kernel prameters.

Based on the mean field approximation, the DenseCRF can be efficiently solved by an iterative message passing algorighm for approximate inference. Detail procedure and derivation is refered to DenseCRF.

Instead of computing the exact distribution $P(\mathbf{X})$, the mean field approximation computes a factored distribution $Q(\mathbf{X})$ that minimizes the KL-divergence $\mathbf{D}(Q(\mathbf{X}) \| P(\mathbf{X}|\mathbf{I}))$ among all distributions Q that can be expressed as a product of independent marginals,

$$P(\mathbf{X}) = \frac{1}{Z(\mathbf{X})} \tilde{P}(\mathbf{X}) = \frac{1}{Z(\mathbf{X})} \exp\{-E(\mathbf{X})\} = \frac{1}{Z(\mathbf{X})} \exp\left\{-\sum_{i=1}^{N} \psi_u(x_i) - \sum_{i<j} \psi_p(x_i, x_j)\right\}$$

$$Q(\mathbf{X}) = \prod_i Q_i(X_i) = Q_1(X_1) Q_2(X_2) \cdots Q_N(X_N)$$

Note that the $Q_i(X_i)$ is a probility distribution, then we

$$Q_i(X_i) \geq 0$$
$$\sum_{l=1}^{L} Q_i(X_i) = 1$$

From the KL-divengence as follows:

$$\begin{aligned}
\mathbf{D}(Q \| P) &= \sum_{\mathbf{x}} Q(\mathbf{x}) \log\left(\frac{Q(\mathbf{x})}{P(\mathbf{x})}\right) \\
&= \sum_{\mathbf{x}} Q(\mathbf{x}) \log Q(\mathbf{x}) - \sum_{\mathbf{x}} Q(\mathbf{x}) \log P(\mathbf{x}) \\
&= \sum_{\mathbf{x}} Q(\mathbf{x}) \log Q(\mathbf{x}) - \sum_{\mathbf{x}} Q(\mathbf{x}) \log\left(\frac{\tilde{P}(\mathbf{x})}{Z}\right) \\
&= \sum_{\mathbf{x}} Q(\mathbf{x}) \log Q(\mathbf{x}) - \sum_{\mathbf{x}} Q(\mathbf{x}) \log \tilde{P}(\mathbf{x}) + \sum_{\mathbf{x}} Q(\mathbf{x}) \log Z \\
&= \sum_{\mathbf{x}} Q(\mathbf{x}) \log Q(\mathbf{x}) - \sum_{\mathbf{x}} Q(\mathbf{x}) \log \tilde{P}(\mathbf{x}) + \log Z \\
&= \sum_{\mathbf{x}} Q(\mathbf{x}) \log Q(\mathbf{x}) - \sum_{\mathbf{x}} Q(\mathbf{x}) \log \tilde{P}(\mathbf{x}) + Const
\end{aligned}$$

We fixed other variables and solve one variable, say $Q_i(X_i)$, then we have

$$\begin{aligned}\mathbf{D}(Q\|P) &= \sum_{\mathbf{x}} Q(\mathbf{x})\log Q(\mathbf{x}) - \sum_{\mathbf{x}} Q(\mathbf{x})\log \tilde{P}(\mathbf{x}) \\ &= D_1 - D_2\end{aligned}$$

$$\begin{aligned}D_1 &= \sum_{\mathbf{x}} Q(\mathbf{x})\log Q(\mathbf{x}) \\ &= \sum_{\mathbf{x}_{-i}}\sum_{x_i} Q(\mathbf{x}_{-i}) Q_i(x_i)\log Q(\mathbf{x}_{-i}) Q_i(x_i) \\ &= \sum_{\mathbf{x}_{-i}}\sum_{x_i} Q(\mathbf{x}_{-i}) Q_i(x_i)\log Q(\mathbf{x}_{-i}) + \sum_{\mathbf{x}_{-i}}\sum_{x_i} Q(\mathbf{x}_{-i}) Q_i(x_i)\log Q_i(x_i) \\ &= \sum_{\mathbf{x}_{-i}} Q(\mathbf{x}_{-i})\log Q(\mathbf{x}_{-i})\sum_{x_i} Q_i(x_i) + \sum_{x_i} Q_i(x_i)\log Q_i(x_i)\sum_{\mathbf{x}_{-i}} Q(\mathbf{x}_{-i}) \\ &= \sum_{\mathbf{x}_{-i}} Q(\mathbf{x}_{-i})\log Q(\mathbf{x}_{-i}) + \sum_{x_i} Q_i(x_i)\log Q_i(x_i) \\ &= Const1 + \sum_{x_i} Q_i(x_i)\log Q_i(x_i)\end{aligned}$$

$$\begin{aligned}D_2 &= \sum_{\mathbf{x}} Q(\mathbf{x})\log \tilde{P}(\mathbf{x}) \\ &= -\sum_{\mathbf{x}_{-i}}\sum_{x_i} Q(\mathbf{x}_{-i}) Q_i(x_i) E(\mathbf{x}) \\ &= -\sum_{x_i} Q_i(x_i)\sum_{\mathbf{x}_{-i}} Q(\mathbf{x}_{-i})\left\{\psi_u(x_i) + \sum_{j\neq i}\psi_p(x_i,x_j) + \sum_{k,k\neq i}\left[\psi_u(x_k) + \sum_{j\neq k}\psi_p(x_k,x_j)\right]\right\} \\ &= -\sum_{x_i} Q_i(x_i)\sum_{\mathbf{x}_{-i}} Q(\mathbf{x}_{-i})\left[\psi_u(x_i) + \sum_{j\neq i}\psi_p(x_i,x_j)\right] \\ &\quad -\sum_{x_i} Q_i(x_i)\sum_{\mathbf{x}_{-i}} Q(\mathbf{x}_{-i})\left\{\sum_{k,k\neq i}\left[\psi_u(x_k) + \sum_{j\neq k}\psi_p(x_k,x_j)\right]\right\} \\ &= -\sum_{x_i} Q_i(x_i)\sum_{\mathbf{x}_{-i}} Q(\mathbf{x}_{-i})\left[\psi_u(x_i)\right] - \sum_{x_i} Q_i(x_i)\sum_{\mathbf{x}_{-i}} Q(\mathbf{x}_{-i})\left[\sum_{j\neq i}\psi_p(x_i,x_j)\right] \\ &\quad -\sum_{\mathbf{x}_{-i}} Q(\mathbf{x}_{-i})\left\{\sum_{k,k\neq i}\left[\psi_u(x_k) + \sum_{j\neq k}\psi_p(x_k,x_j)\right]\right\} \\ &= -\sum_{x_i} Q_i(x_i)\psi_u(x_i) - \sum_{x_i} Q_i(x_i)\sum_{\mathbf{x}_{-i}} Q(\mathbf{x}_{-i})\sum_{j\neq i}\psi_p(x_i,x_j) \\ &= -\sum_{x_i} Q_i(x_i)\psi_u(x_i) - \sum_{x_i} Q_i(x_i)\sum_{j\neq i}\mathbf{E}_{U_j\sim Q_j}\psi_p(x_i,U_j)\end{aligned}$$

Then the Lagrange equations is

$$\mathcal{L} = \mathbf{D}(Q\|P) + \sum_{i=1}^{N}\left(\lambda_i\left[\sum_{x_i} Q_i(x_i) - 1\right]\right)$$

then set the gradient w.r.t $Q_i(x_i)$ to zero, as

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial Q_i(x_i)} &= \log Q_i(x_i) + 1 \\ &\quad +\psi_u(x_i) + \sum_{j\neq i}\mathbf{E}_{U_j\sim Q_j}\left[\psi_p(x_i,U_j)\right] \\ &\quad +\lambda_i \\ &= 0\end{aligned}$$

let

$$\begin{aligned} K_l &= \psi_u(x_i = l) + \sum_{j\neq i} \mathbf{E}_{U_j\sim Q_j}\left[\psi_p(x_i = l, U_j)\right] \\ \log Q_i(x_i = l) &= -1 - \lambda_i - K_l \end{aligned}$$

$$Q_i(x_i = l) = \exp(-1-\lambda_i)\exp(-K_l)$$

then

$$\begin{aligned} Q_i(x_i = 1) &= \exp(-1-\lambda_i)\exp(-K_l) \\ Q_i(x_i = 2) &= \exp(-1-\lambda_i)\exp(-K_2) \\ &\vdots \\ Q_i(x_i = L) &= \exp(-1-\lambda_i)\exp(-K_L) \end{aligned}$$

then

$$1 = \sum_{x_i} Q_i(x_i = l) = \exp(-1-\lambda_i)\sum_{l=1}^{L}\exp(-K_l)$$

then

$$\exp(-1-\lambda_i) = \frac{1}{\sum_{l=1}^{L}\exp(-K_l)}$$

$$\begin{aligned} Q_i(x_i = l) &= \exp(-1-\lambda_i)\exp(-K_l) \\ &= \frac{\exp(-K_l)}{\sum_{l=1}^{L}\exp(-K_l)} \\ &= \frac{1}{Z_i}\exp\left\{-\psi_u(x_i = l) - \sum_{j\neq i}\mathbf{E}_{U_j\sim Q_j}\left[\psi_p(x_i = l, U_j)\right]\right\} \end{aligned}$$

$$Z_i = \sum_{l=1}^{L}\exp\left\{-\psi_u(x_i = l) - \sum_{j\neq i}\mathbf{E}_{U_j\sim Q_j}\left[\psi_p(x_i = l, U_j)\right]\right\}$$

On the other hand, we have

$$\psi_p(x_i, x_j) = \mu(x_i, x_j)\sum_{m=1}^{K} w^{(m)} k^{(m)}(\mathbf{f}_i, \mathbf{f}_j)$$

then we have

$$\begin{aligned} Q_i(x_i = l) &= \frac{1}{Z_i}\exp\left\{-\psi_u(x_i) - \sum_{j\neq i}\mathbf{E}_{U_j\sim Q_j}\left[\mu(l, U_j)\sum_{m=1}^{K} w^{(m)} k^{(m)}(\mathbf{f}_i, \mathbf{f}_j)\right]\right\} \\ &= \frac{1}{Z_i}\exp\left\{-\psi_u(x_i) - \sum_{m=1}^{K} w^{(m)}\sum_{j\neq i}\mathbf{E}_{U_j\sim Q_j}\left[\mu(l, U_j)\, k^{(m)}(\mathbf{f}_i, \mathbf{f}_j)\right]\right\} \\ &= \frac{1}{Z_i}\exp\left\{-\psi_u(x_i) - \sum_{m=1}^{K} w^{(m)}\sum_{j\neq i}\sum_{l'=1}^{L} Q_j(l')\,\mu(l, l')\, k^{(m)}(\mathbf{f}_i, \mathbf{f}_j)\right\} \\ &= \frac{1}{Z_i}\exp\left\{-\psi_u(x_i) - \sum_{l'=1}^{L}\mu(l, l')\sum_{m=1}^{K} w^{(m)}\sum_{j\neq i} k^{(m)}(\mathbf{f}_i, \mathbf{f}_j)\, Q_j(l')\right\} \end{aligned}$$